(1) (A) False.

Let A be any non-measurable subset of $\mathbb{R}$.

Then both A & A^c are uncountable.

we can find $f_1 : A \to (0,\infty)$ & $f_2 : A^c \to (-\infty,0)$ $\ni$ f_1 & f_2 are bijections.

Then define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} f_1(x) & \text{if } x \in A \\ f_2(x) & \text{if } x \in A^c \end{cases}$

$\therefore f^{-1}(\{a\})$ is a Lebesgue set, $\forall a \in \mathbb{R}$ ($\because f^{-1}(\{a\})$ is a singleton set)

but $f^{-1}((0,\infty)) = A$ is not measurable

$\therefore f$ is not Lebesgue measurable.

(B) False.

Let A be any Lebesgue measurable set which is not Borel measurable.

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \chi_A(x)$

$f^{-1}(\{a\})$ is a Lebesgue set, $\forall a \in \mathbb{R}$. but f is not Borel measurable $\because f^{-1}((\frac{1}{2},\infty)) \notin \mathcal{B}(\mathbb{R})$.

(C) True.

Any function $f : (\mathbb{R}, 2^{\mathbb{R}}) \to \mathbb{R}$ is measurable.

---

(2) (A) & (B) False.

Let $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = [\cos(2\pi n! x)]^{2n}$, $\forall n \in \mathbb{N}$

Then $f_n \in C[0,1]$, $\forall n \Rightarrow f_n$ is Riemann integrable, $\forall n$.

& $\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \in \mathbb{Q}^c \cap [0,1] \end{cases}$

f is not Riemann integrable.

(C) True.

Since, f_n's are Rieman integrable, f_n's are Lebesgue measurable & limit of a sequence of Lebesgue measurable functions is Lebesgue measurable.

---

(3) (A) False.

Let $f_n:\mathbb{R}\to\mathbb{R}$ by $f_n(x) = \begin{cases} nx(1-x^2)^n & \text{if } x\in[0,1] \\ 0 & \text{otherwise} \end{cases}$, $\forall n\in\mathbb{N}$

f_n's are continuous & f_n converges pointwise to $f\equiv 0$.

$$\int_0^1 f_n(x)\,dx = \int_0^1 nx(1-x^2)^n\,dx = \int_0^1 \frac{d}{dx}\left[(1-x^2)^{n+1}\right]\cdot\left(\frac{-n}{2(n+1)}\right)dx$$

$$= \frac{n}{2(n+1)}$$

$\therefore \lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \frac{1}{2} \neq 0 = \int_0^1 f(x)\,dx$

(B) True

Given $0\le f_n(x)\le f(x)$, $\forall n\in\mathbb{N}$, $\forall x\in\mathbb{R}$.

Since $f:\mathbb{R}\to\mathbb{R}$ is continuous, $\int_a^b f(x)\,dx<\infty$, $\forall a,b\in\mathbb{R}$

$\therefore$ By Dominated Convergence Theorem (DCT),

$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$, $\forall a,b\in\mathbb{R}$.

(C) True.

~~when~~ $a\in\mathbb{R}$, $\int_a^\infty e^{-x}\,dx = e^{-a}<\infty$

$\therefore$ By DCT, $\lim_{n\to\infty}\int_a^\infty f_n(x)\,dx = \int_a^\infty f(x)\,dx$

(D) False.

Define $f_n:\mathbb{R}\to\mathbb{R}$ by $f_n(x) = \begin{cases} 2(x-n) & \text{if } n\le x\le n+\frac{1}{2} \\ 2(n+1-x) & \text{if } n+\frac{1}{2}<x\le n+1 \\ 0 & \text{otherwise} \end{cases}$

For $a\in\mathbb{R}$, $\int_a^\infty f_m(x)\,dx = \frac{1}{2}$, for all $m\in\mathbb{N}\ni m\ge a$.

$\therefore \lim_{n\to\infty}\int_a^\infty f_n(x)\,dx = \frac{1}{2}$, $\forall a\in\mathbb{R}$. But, here $f(x)=0, \forall x\in\mathbb{R}$.

$\therefore \lim_{n\to\infty}\int_a^\infty f_n(x)\,dx = \frac{1}{2} \neq 0 = \int_a^\infty f(x)\,dx$, $\forall a\in\mathbb{R}$.

---

④

(A) False.

Take $f = \chi_{[0,1]} - \chi_{[1,2]}$

$\int_{\mathbb{R}} f \, dm = 0$ but $f \equiv 1$ on $[0,1]$.

(B) True.

$E = \{x \in \mathbb{R} : |f(x)| > 0\}$

$E = \bigcup_{n=1}^{\infty} E_n$, where $E_n = \{x \in \mathbb{R} : |f(x)| \geq 1/n\}$

Since $\int_{E_n} |f| \, dm \leq \int_{\mathbb{R}} |f| \, dm = 0$

$\Rightarrow 1/n \int_{E_n} dm \leq \int_{E_n} |f| \, dm = 0$.

$\Rightarrow m(E_n) = 0$. $\Rightarrow m(E) = 0$.

$\Rightarrow f = 0$ a.e on $\mathbb{R}$.

(C) True.

$E = \{x \in \mathbb{R} : f(x) \neq 0\}$.

$E = \bigcup_{n=1}^{\infty} (E_n \cup E_{-n})$ where $E_n = \{x \in \mathbb{R} : f(x) \geq 1/n\}$.

$E_{-n} = \{x \in \mathbb{R} : f(x) \leq -1/n\}$.

If we proceed like in option (B), we can prove

$m(E_n) = m(E_{-n}) = 0$, $\forall n \in \mathbb{N}$.

$\Rightarrow m(E) = 0$

$\Rightarrow f = 0$ a.e on $\mathbb{R}$.

---

(5)

(A) True.

$f:[0,1]\to\mathbb{R}$ by $f(x)=\begin{cases}\sqrt{x} & \text{if } x\in Q^c\cap[0,1]\\ 0 & \text{if } x\in Q\cap[0,1]\end{cases}$

$\therefore f(x)=\chi_{Q^c\cap[0,1]}(x).\sqrt{x}$

Since, $\chi_{Q^c[0,1]}(x)$ & $\sqrt{x}$ are measurable and product of two measurable functions is measurable.

f is measurable.

(B) True.

Since, f is equal to $\sqrt{x}$ a.e on $[0,1]$

$$\int_{[0,1]} f\,dm = 2/3$$

$\Rightarrow f$ is Lebesgue integrable.

(C) False.

Since, ~~th~~ upper sum of f is $2/3$ and lower sum of f is 0.

f is not Riemann integrable.

Scanned with CamScanner

(6) (A) True.

$E \subseteq \mathbb{R}$, χ_E is measurable.

Then, $\chi_E^{-1}(\{1\}) = E$ is a Lebesgue set.

(B) True.

Arguement is similar to the option (A).

---

(7) (A) True.

Let $a_1, a_2 \ldots a_n$ be such finitely many points of $f(a_i) \neq g(a_i)$

Consider the partition which always include these points.

$\therefore$ Then g is Riemann integrable

(B) False.

$f(x) = 1$, $\forall x \in [0,1]$.

& $g(x) = \chi_{\mathbb{Q}^c \cap [0,1]}(x)$

$g = f$ except on countably many points.

But g is not Riemann integrable.

(C) True.

Since $\int f \, dm = \int g \, dm$

---

(8) (A) NOT σ-finite.

$[0,1] = \bigcup_{n=1}^{\infty} X_n$, atleast one X_m should contain uncountably many elements of $[0,1]$.

(B) NOT σ-finite.

(C) μ is an σ-finite measure.

$\mathbb{R} = \bigcup_{n=0}^{\infty} X_n$, $X_0 = \mathbb{R} \setminus \mathbb{Q}$ & $X_n = \{r_n\}$, where (r_n) is an enumeration of $\mathbb{Q}$. for $n \in \mathbb{N}$

$\mu(X_n) \leq 1$, $\forall n \in \mathbb{N} \cup \{0\}$.

(D) μ is an σ-finite measure.

$\mathbb{R} = \bigcup_{n=0}^{\infty} X_n$, $X_0 = (\mathbb{R} \setminus \mathbb{Z}) \cup \{0\}$, $X_n = \{n, -n\}$, for $n \in \mathbb{N}$.

$\mu(X_n) \leq 2$, $\forall n \in \mathbb{N} \cup \{0\}$.

---

(9)

(A) NOT well defined.

(B) +ve linear functional.

Since, $f \in C_c(\mathbb{R})$, ~~on~~ $f(n)$ is non-zero only for finitely many $n \in \mathbb{N}$. T is well defined

$\therefore$ when $f(x) \geq 0$, $T(f) \geq 0$.

(C) +ve linear functional.

Since, $f \in C_c((0,\infty))$, $f(1/n)$ is non zero only for finitely many $n \in \mathbb{N}$. T is well defined.

$\therefore$ when $f(x) \geq 0$, $T(f) \geq 0$.

---

(10) (A) +ve linear functional.

$f \in C_c(X)$, then $f(1/k)$ & $f(-1/k)$ are none zero only for finitely many $k \in \mathbb{N}$. $\Rightarrow$ T is well defined.

$\therefore$ when $f \geq 0$, $T(f) \geq 0$.

(B) +ve linear functional

$f \in C_c(X)$, $f(k)$ & $f(-k)$ are non zero only for finitely many $k \in \mathbb{N}$ $\Rightarrow$ T is well defined.

& when $f \geq 0$ $T(f) \geq 0$.

(C) +ve linear functional.

$f \in C_c(X)$, $f(k + 1/k)$ is non zero only for finitely many $k \in \mathbb{N}$. $\Rightarrow$ T is well defined.

& when $f \geq 0$, $T(f) \geq 0$.
